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Comparison of computational times and accuracy for finite element solution of Navier–Stokes equations and Ladyzhenskaya equations

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SUMMARY

In this paper we consider a pure-streamfunction equation of the Ladyzhenskaya equations. For certain values of the parameters of the equation, the studied equation becomes identical to the pure-streamfunction equation of the Navier–Stokes equations. A weak form, a finite element method approximation procedures, and an iterative method for solving the discrete nonlinear problems are provided. Using the Bogner–Fox–Schmidt element, the steady 2-D incompressible flow in a driven cavity is solved using a grid mesh of 16×16. Streamfunction contours are also displayed showing the main features of the flow. Copyright © 2007 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Understanding flow is central to many important problems in engineering. However, in many situations it is still not clear which models are the most appropriate. The Navier–Stokes equations (NSE) are generally accepted as providing an accurate model for the incompressible motion of viscous fluids in practical situations. This research will consider one model introduced by Ladyzhenskaya [1, 2]. The following paragraph, summarized from [3], addresses the reasons for choosing the Ladyzhenskaya model and the attractions of the streamfunction formulation. From a modeling stand point, the NSE are a special case of the Ladyzhenskaya equations (LE). This leads to the conclusion that any flow that can be accurately described by solutions of the NSE can

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be at least as accurately described by solutions of the LE. For certain values of the parameter q, the LE considered here are identical to the Smagorinsky model [4]. Ladyzhenskaya has shown [1, 2] that solutions of the equations in the non-stationary case and in three space dominions are globally unique in time. The analogous result for the NSE has not been proved and is believed not to be true. The condition derived in [3] which guarantees the uniqueness of the solutions of the stationary LE model is, in some sense, less pessimistic than the analogous condition for the NSE. Also, the analogous condition for the stationary Ladyzhenskaya model [5, 2] generally guarantees uniqueness for the higher values of the Reynolds number than that predicted for the Navier–Stokes model. This research also has the advantage of using the streamfunction formulation of LE. The attractions of the streamfunction formulation are that the incompressibility constraint is automatically satisfied, the pressure is not present in the weak form, and there is only one scalar unknown to solve for.

2. MODEL EQUATION

The model we work with is as follows: consider the motion of stationary ideal incompressible viscous fluids in a bounded domain Ω in R^2 with Lipshitz boundary $\partial\Omega$, and let u denote the velocity field, p the pressure, ϕ the streamfunction, and f the body force per unit mass. Then the model is given by [SL]

$$\partial_{xx}(A\psi_{xx}) + 2\partial_{xy}(A\psi_{xy}) + \partial_{yy}(A\psi_{yy}) - \psi_y \Delta\psi_x + \psi_x \Delta\psi_y = f_{2,x} - f_{1,y} \quad \text{in } \Omega \tag{1}$$

$$\psi = \frac{\partial\psi}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{2}$$

where $A = \varepsilon_0 + \varepsilon_1 \|\boldsymbol{\Delta}\psi\|^{q-2}$, $\boldsymbol{\Delta}\psi = [\psi_{xx}, \psi_{xy}, \psi_{yx}, \psi_{yy}]^{\mathrm{T}}$, and $\|\boldsymbol{\Delta}\psi\| = [\psi_{xx}^2 + 2\psi_{xy}^2 + \psi_{yy}^2]^{1/2}$ with $\varepsilon_0 = (1/Re)$, where Re is the Reynolds number, $\varepsilon_1 > 0$ and $q - 2 > 0$. If we set $\varepsilon_1 = 0$, equation [SL] reduces to the streamfunction form of the NSE. Finite element analysis of this form can be found in [6]. Fairag [5, 7, 8] studied the two-level finite element analysis and some computational aspects.

[SL] is derived from the LE in the velocity–pressure form:

$$-\nabla(\hat{A}_1(u)\nabla u) + (u \cdot \nabla)u + \nabla p = f \tag{3}$$

Many researchers in LES prefer the use of the symmetric part of the gradient $\nabla^{\mathrm{s}} u = (\nabla u + \nabla u^{\mathrm{T}})/2$. In this case, Equation (3) becomes

$$-\nabla(\tilde{A}(u)\nabla^{\mathrm{s}} u) + (u \cdot \nabla)u + \nabla p = f \tag{4}$$

where $\tilde{A}(u) = \varepsilon_0 + \varepsilon_1 \|\nabla^{\mathrm{s}} u\|_{\mathrm{F}}^{q-2}$, $\tilde{A}_1(u) = \varepsilon_0 + \varepsilon_1 \|\nabla u\|^{q-2}$, as $\|\cdot\|_{\mathrm{F}}$ is the Frobenius norm defined by: for all $V \in R^2$, $\|V\|_{\mathrm{F}} = (\sum_{i,j=1}^{2} V V^{\mathrm{T}})^{1/2}$. With Korns inequality, all results proven for Equation (3) can be extended to Equation (4) immediately. The LE, (3) have been proposed in [1, 2]. Finite element error analysis of this model was carried out in Du and Gunzburger [3, 9] under a global uniqueness (small data) condition. Layton provided [10] an error analysis for the high Re number; also he provided a formula for choosing q and ε_1 so that one can construct a higher-order method that is just as stable as a first-order upwind method. Iterative method for solving the discrete nonlinear problems (3) is given in [9].

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3. TWO WEAK FORMULATIONS

In this section, we introduce two weak formulations for the streamfunction equation of the LE. We follow the notation of [11]. For each $\phi \in H^1(\Omega)$, define $\operatorname{curl}\phi = (\phi_y, -\phi_x)^{\mathrm{T}}$. For each $\mathbf{u} = (u_1, u_2)^{\mathrm{T}} \in [H^1(\Omega)]^2$, define $\operatorname{curl}\mathbf{u} = (\partial u_2/\partial x) - (\partial u_1/\partial y)$. Let $W^{m,p}(\Omega)$ be the Sobolev space associated with the norm $\|\phi\|_{m,p,\Omega} := (\sum_{|\alpha|\leqslant m} \int_\Omega |\partial^\alpha \phi(x)|^p \,\mathrm{d}\Omega)^{1/p}$, $p<\infty$. We define for $q>0$, $W_0^{2,q}(\Omega) :=$ completion of $C_0^\infty(\Omega)$ in the $W^{2,q}$-norm.

We now present the weak formulation for problem [SL] which can be obtained through the standard procedure, e.g. multiplying the original equation by test functions and integrating by parts [WSL1]:

$$\begin{gathered}\text{Find } \psi \in W_0^{2,q}(\Omega) \text{ such that for all } \phi \in W_0^{2,q}(\Omega)\\ a_1(\psi, \phi) + \hat{a}_1(\psi, \psi, \phi) + b(\psi; \psi, \phi) = (\mathbf{f}, \operatorname{curl}\phi)\end{gathered} \tag{5}$$

where

$$a_1(\psi, \phi) = \varepsilon_0 \int_\Omega \Delta\psi \Delta\phi \,\mathrm{d}x\,\mathrm{d}y, \quad \hat{a}_1(\psi, \xi, \phi) = \varepsilon_1 \int_\Omega \|\boldsymbol{\Delta}\psi\|^{q-2} \boldsymbol{\Delta}\xi \boldsymbol{\Delta}\phi \,\mathrm{d}x\,\mathrm{d}y$$

$$b(\psi; \xi, \phi) = \int_\Omega \Delta\psi(\xi_y \phi_x - \xi_x \phi_y)\,\mathrm{d}x\,\mathrm{d}y, \quad (\mathbf{f}, \operatorname{curl}\phi) = \int_\Omega \mathbf{f} \cdot \operatorname{curl}\phi \,\mathrm{d}x\,\mathrm{d}y$$

Similarly, multiplying the streamfunction equation of (4) by test function and integrating by parts give [WSL2]

$$\begin{gathered}\text{Find } \psi \in W_0^{2,q}(\Omega) \text{ such that for all } \phi \in W_0^{2,q}(\Omega)\\ a_2(\psi, \phi) + \hat{a}_2(\psi, \psi, \phi) + b(\psi; \psi, \phi) = (\mathbf{f}, \operatorname{curl}\phi)\end{gathered} \tag{6}$$

where

$$a_2(\psi, \phi) = \varepsilon_0 \int_\Omega [2\psi_{xy}\phi_{xy} + \frac{1}{2}(\psi_{yy} - \psi_{xx})(\phi_{yy} - \phi_{xx})]\,\mathrm{d}\Omega$$

$$\hat{a}_2(\psi, \psi, \phi) = \varepsilon_1 \int_\Omega \|\Delta_{\mathrm{s}}\psi\|_{\mathrm{F}}^{q-2} \left[2\psi_{xy}\phi_{xy} + \frac{1}{2}(\psi_{yy} - \phi_{yy})(\phi_{yy} - \phi_{xx})\right] \mathrm{d}\Omega$$

$$\|\Delta_{\mathrm{s}}\phi\|_{\mathrm{F}}^2 = 2\phi_{xy}^2 + \frac{1}{2}(\phi_{yy} - \phi_{xx})^2$$

Existence and uniqueness and finite element error analysis of the weak form [WSL1] were carried out in [12]. [WSL1] has a unique solution under a certain condition depending on its parameters. This uniqueness property has been proved in [12]. In this paper, we focus our attention on finite element approximations of the model problems [WSL1] and [WSL2] described above.

4. DISCRETIZATION AND ITERATIVE METHOD

Let Ω^h be a regular finite element triangulation where h is a discretization parameter that tends to zero. We define a finite-dimensional space X^h such that $X^h \subset W_0^{2,q}(\Omega)$. Then, we approximate

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Table I. Accuracy of finite elements for the streamfunction form.

Finite element	Argyis	Clough–Tocher	Bogner–Fox–Schmidt	Bicubic spline
$\|\psi-\psi^h\|_2=O(h^s)$	$s=4$	$s=2$	$s=2$	$s=2$

[WSL1] and [WSL2] by the following the discrete problem:

$$\begin{gathered}\text{Find } \psi^h\in X^h \text{ such that for all } \phi^h\in X^h\\ a_i(\psi^h,\phi^h)+\hat{a}_i(\psi^h,\psi^h,\phi^h)+b(\psi^h;\psi^h,\phi^h)=(\mathbf{f},\operatorname{curl}\phi^h)\end{gathered}\tag{7}$$

where in (7) $i=1,2$. Existence and uniqueness of the solution to (7) can be found in [12]. The inclusion $X^h\subset W_0^{2,q}(\Omega)$ requires the use of finite-element functions that are continuously differentiable over Ω. Argyis triangle, Clough–Tocher triangle, Bogner–Fox–Schmidt rectangle, and Bicubic spline rectangle [8] are examples of finite-element spaces for the streamfunction formulation of the LE. In [12], we established the error bound given in the following theorem. This theorem and its proof can also be found in [12].

Theorem 4.1
Let $X^h\subset W_0^{2,q}(\Omega)$ be a finite element space. Let ψ be the solution to [WSL1] and ψ^h be the solution to (7). Then $\|\psi-\psi^h\|_{H_0^2(\Omega)}\leqslant C(Re)\inf_{\psi^h\in X^h}\|\psi-\psi^h\|_{H_0^2(\Omega)}$, for h sufficiently small.

As an example, if the Bogner–Fox–Schmidt rectangles are used, then there exist a positive constant C such that $|\psi-\psi^h|_2\leqslant Ch^2$. For each of the elements mentioned above, Table I shows the error estimates.

Now, we will describe an iterative method for the nonlinear system resulting from the discretization. First, we linearize the added nonlinear term and then solve the nonlinear system of equations. Let $\psi^{(0)}\in X^h$ be given, then we define the sequence $\psi^{(n)}\in X^h$ for $n=1,2,3,.,$ to be the solution of the following nonlinear discrete system:

$$\begin{gathered}\text{Find } \psi^{(n)}\in X^h \text{ such that for all } \phi^h\in X^h\\ a_i(\psi^{(n)},\phi)+\hat{a}_i(\psi^{(n-1)},\psi^{(n)},\phi)+b(\psi^{(n)};\psi^{(n)},\phi)=(\mathbf{f},\operatorname{curl}\phi)\end{gathered}\tag{8}$$

The resulting system from (8) is a nonlinear system and if we use Newton's method to solve this nonlinear system, then the resulting matrix from each iteration is sparse and nonsymmetric, whose symmetric part is positive definite. The suggested linear solver for such a system is any conjugate gradient-alike method. It is known that the convergence is very rapid when the errors are small. Choosing $\psi^{(n-1)}$ to be the initial guess in solving the nonlinear system (8) is considered to be a good choice because as n increases $\psi^{(n)}$ tends closer and closer to the exact solution. For $n=1$, we choose $\psi^{(0)}=0$ which leads to solve the NSE. The requirement $\|\text{residual}^{(n+1)}\|\leqslant\|\text{residual}^{(n)}\|,\ n=0,1,\ldots,$ is imposed on the method.

5. COMPUTATIONAL EXPERIMENTS

We consider the driven cavity problem in the 2-D box $[0, 1] \times [0, 1]$ with no-slip boundary conditions, i.e. $u_1 = u_2 = 0$ in all boundaries except $y = 1$, where $u_1 = 1$. This problem has been studied and addressed by many researchers [13]. The numerical computation in this section was obtained using a Compaq nc8230 with Intel Mobile CPU 1.86 GHz running Windows XP. Bogner–Fox–Schmit elements are used with 8×8 and 10×10 grid points.

5.1. Different Reynolds numbers

We choose $\varepsilon_1 = 1.0\text{e} - 40$ and $q = 4$. We compute an approximate solution using the weak form [WSL2] for $Re = 1$, 10, 30, 100, and 150. Also, we compute an approximate solution for the NSE with $Re = 150$. Figure 1 displays streamfunction contours. Our computations show that we obtain a stable approximation of the solution for the LE. The top right corner shows that the contours are very smooth. In the bottom right corner, the number of vortices and the size of the vertices increase as the Reynolds number increases.

5.2. Different values for q

We solve the LE using (6) with 10×10 grid points on $[0, 1] \times [0, 1]$ with fixed Reynolds number $Re = 10$, $\varepsilon_1 = 1.0\text{e} - 40$, and $q = 2, 4$, and 6. Also, we solve the NSE with $Re = 10$ on the same mesh. Then, we compute $\|\psi_{\text{L}}^h - \psi_{\text{q}}^h\|$ where ψ_{q}^h is the approximate solution of the LE and ψ_{ns}^h is the approximate solution of the NSE. Then, we compute $\|\cdot\|_{0,\Omega}$ and $\|\cdot\|_{1,\Omega}$. Table II shows the result of these computations.

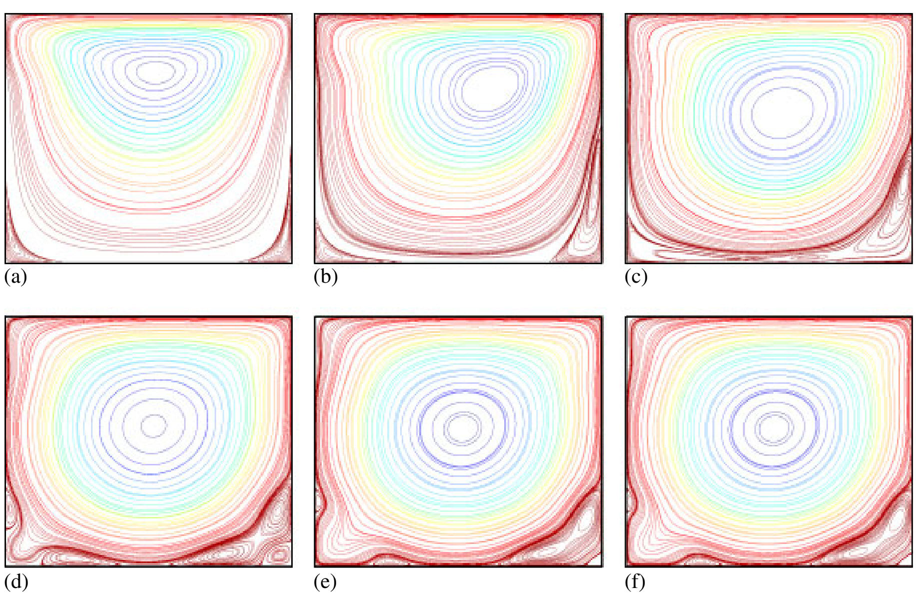

Figure 1. Contours for different values of Re: (a) $Re = 1$; (b) $Re = 10$; (c) $Re = 30$; (d) $Re = 100$; (e) $Re = 150$; and (f) NSE with $Re = 150$.

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Table II. Differences between the approximate solution of the streamfunction using LE and NSE.

	$\|\psi_2^h - \psi_{ns}^h\|_s$	$\|\psi_4^h - \psi_{ns}^h\|_s$	$\|\psi_6^h - \psi_{ns}^h\|_s$	$\|\psi_2^h - \psi_4^h\|_s$
$s=0$	4.257E−5	4.257E−5	9.0914E−10	9.136E−11
$s=1$	7.469E−5	7.469E−5	2.0000E−09	1.1343E−09

Table III. The CPU-time ratio to the NSE [WSL1].

	NSE	LE-12	LE-11
WSL1	1	2.04	6.01
WSL2	1.02	2.79	12.79

Table IV. Cases of divergence.

	NSE [WSL1]	NSE [WSL2]	LE [WSL1]	LE [WSL2]
$Re=170$	conv	div	conv	conv
$Re=175$	div	div	conv	conv
$Re=180$	div	div	conv	div

5.3. CPU-time comparisons

We solve the LE and NSE equations using [WSL1] and [WSL2] with 10×10 grid points having fixed Reynolds number $Re=10$, $q=4$, and the same initial guess. Table III summarizes the speed comparisons. The second and third rows show the ratio of CPU time to that of NSE using [WSL1]. Thus, NSE using [WSL1] is 6.01 times faster than LE with $\varepsilon=1\mathrm{e}-11$ using [WSL1], and 2.79 times faster than LE with $\varepsilon=1\mathrm{e}-12$ using [WSL2]. In general, using [WSL1] is faster than using [WSL2].

5.4. Cases of divergence

We solve the NSE and LE using [WSL1] and [WSL2] with 8×8 grid points and $q=4$, $\varepsilon=1\mathrm{E}-20$, $1\mathrm{E}-22$, $1\mathrm{E}-23$, and we use the solution of the NSE with $Re=100$ as an initial guess. Table IV summarizes these cases of divergence.

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